THE HYPERPLANE MEAN OF A NON-NEGATIVE SUBHARMONIC FUNCTION

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Introduction

Let m be a positive integer and R^{m+1} be the euclidean space of dimension m+1; O denotes the origin, and an arbitrary point of R^{m+1} is represented by

$$M = (x,y) = (x_1,x_2,\dots,x_m,y).$$

We put

$$X^2 = |X|^2 = x_1^{\,2} + \dots + x_m^{\,2},\ dX = dx_1\dots dx_m.$$

For each real number a, let D_a be the open half-space

$$D_a = \{M \in R^{m+1} \mid y > a\}.$$

When a = 0, D_a is simply replaced by D. In D, the Poisson kernel P(X,y) is given by

$$P(X,y) = 2s_{m+1}^{-1}y(X^2+y^2)^{-\frac{1}{2}(m+1)}$$

where s_{m+1} denotes the surface-area of the unit sphere in R^{m+1}

It is well-known that

$$\int_{R^m} P(X,y)dX = 1.$$

Let f be a function defined at least on the hyperplane given by the equation y = a, such that the function

$$X \longmapsto (X^2+1)^{-\frac{1}{2}(m+1)}f(X,a)$$

is Lebesgue integrable in R^m. The Poisson integral I_f^a in D_a of

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the restriction of f to the hyperplane is given by

$$I_f^a(X,y) = \int_{R^m} P(X-Z,y-a)f(Z,a)dZ$$

for all (X,y) belonging to D_a. We note that I_f^a is harmonic in D_a (Flett, [1], Theorem 6).

Let u be a non-negative subharmonic function in D, we define a function $M(u,\cdot)$ on $(0, +\infty)$ by writting

$$M(u,y) = \int_{R^m} u(X,y)dX .$$

The integral being taken in the sense of Lebesgue. Thus $M(u,\cdot)$ exists and is non-negative (the value $+\infty$ being permitted) in $(0, +\infty)$. In 1983, Rippon ([2], Theorem 2) proved the following:

Theorem A. If u is a positive subharmonic function and has a harmonic majorant in D, if $M(u,1) < +\infty$, then either $M(u,\cdot)$ is decreasing in $[1, +\infty)$ or

$$\int_1^{+\infty} \min\,[1, (y/M(u,y))^{1/m}]dy < +\infty .$$

The purpose of this paper is devoted to study Theorem A in the case that u has no harmonic majorant in D. We get the we get the following result.

Theorem. Suppose that

(i) u is a non-negative subharmonic function in D,

(ii) u has no harmonic majorant in $R^m \times (1, +\infty)$,

(iii) $M(u,\cdot)$ is locally bounded in $(0, +\infty)$.

Then

$$\lim_{y \to +\infty} \frac{M(u,y)}{y^{m+1}} = +\infty.$$

<u>Proof</u>. Let $a > 1$, we first prove that

$$\lim_{\substack{|(X,y)| \to +\infty \\ (X,y) \in R^m \times [1,a]}} u(X,y) = 0. \tag{1}$$

We shall work by contradiction. Suppose (1) is not true, then there is a sequence of points $(P_n) = ((X_n, y_n))$ in $R^m \times [1,a]$ such that $|P_n| \to +\infty$ as $n \to +\infty$ and $u(P_n) \geqslant \varepsilon$ for all n. By working with a subsequence, if necessary, we may also suppose that $B(P_n, \frac{1}{2}) \cap B(P_k, \frac{1}{2}) = \Phi$ whenever $n \neq k$ (where $B(P,r)$ is the open ball of center P and of radius r). By the volume mean-value inequality, for each $n \in N$, we have

$$\int_{B(P_n,\frac{1}{2})} u(X,y)dXdy \geqslant v_{m+1}(\tfrac{1}{2})^{m+1} u(P_n) \geqslant v_{m+1}(\tfrac{1}{2})^{m+1}\varepsilon$$

where v_{m+1} is the volume of $B(0,1)$ in R^{m+1}. Hence

$$\int_{R^m \times (\frac{1}{2}, a+\frac{1}{2})} u(X,y)dXdy \geqslant \int_{\bigcup_{n=1}^{\infty} B(P_n,\frac{1}{2})} u(X,y)dXdy$$

$$= \sum_{n=1}^{\infty} \int_{B(P_n,\frac{1}{2})} u(X,y)dXdy$$

$$= +\infty .$$

On the other hand, since $M(u, \cdot)$ is locally bounded in $(0, +\infty)$, we have

$$\int_{R^m \times (\frac{1}{2}, a+\frac{1}{2})} u(X,y)dXdy = \int_{\frac{1}{2}}^{a+\frac{1}{2}} M(u,y)dy < +\infty .$$

Thus we have a contradiction and then (1) is true.

Now let $\eta > 0$. Define I_u in $R^m \times (1, +\infty)$ by

$$I_u(X,y) = \frac{2(y-1)}{s_{m+1}} \int_{R^m} \frac{u(Z,1)dZ}{\{|X-Z|^2 + (y-1)^2\}^{(m+1)/2}}$$

and define J_u in $R^m \times (-\infty, 2\eta + 1)$ by

$$J_u(X,y) = \frac{2(2\eta+1-y)}{s_{m+1}} \int_{R^m} \frac{u(Z,2\eta+1)dZ}{\{|X-Z|^2 + (2\eta+1-y)^2\}^{(m+1)/2}}$$

Thus I_u and J_u are half-space Poisson integrals and $I_u + J_u$ is harmonic in $R^m \times (1,2\eta+1) = \Omega_\eta$ say. We next aim to prove that

(2) $$u \leqslant I_u + J_u \quad \text{in } \Omega_\eta.$$

It follows from (1) and the upper semi-continuity of u that u is bounded on $R^m \times \{1\}$ and $R^m \times \{2\eta+1\}$. Hence there are two decreasing sequences (f_n) and (g_n) of bounded real-valued continuous functions on R^m such that

$$\lim_{n \to +\infty} f_n(Z) = u(Z,1) \quad \text{and} \quad \lim_{n \to +\infty} g_n(Z) = u(Z,2\eta+1)$$

for all Z in R^m. Then, for each n, I_{f_n} is harmonic in $R^m \times (1,+\infty)$ and J_{g_n} is harmonic in $R^m \times (-\infty,2\eta+1)$. Also

$$\lim_{n \to +\infty} (I_{f_n} + J_{g_n}) = I_u + J_u \quad \text{in } \Omega_\eta.$$

Hence, to prove (2), it is enough to show

(3) $$u \leqslant I_{f_n} + J_{g_n} \quad \text{in } \Omega_\eta.$$

Since we have

$$(4)\qquad \lim_{\substack{(X,y)\to(Z,1)\\ (X,y)\in\Omega_\eta}} I_{f_n}(X,y) + J_{g_n}(X,y) = f_n(Z,1) + J_{g_n}(Z,1) \geqslant f_n(Z,1) \geqslant u(Z,1).$$

Similarly, we have

$$(5)\qquad \lim_{\substack{(X,y)\to(Z,2\eta+1)\\ (X,y)\in\Omega_\eta}} I_{f_n}(X,y) + J_{g_n}(X,y) \geqslant u(Z,2\eta+1).$$

Also, by (1),

$$(6)\qquad \lim_{\substack{|(X,y)|\to+\infty\\ (X,y)\in\Omega_\eta}} \inf\,(I_{f_n}(X,y) + J_{g_n}(X,y)) \geqslant 0 = \lim_{\substack{|(X,y)|\to+\infty\\ (X,y)\in\Omega_\eta}} u(X,y).$$

Inequality (3) now follows from (4),(5),(6), and the maximum principle. Hence (2) is true. In particular, for all $X \in R^m$

$$u(X,\eta+1) \leqslant \frac{2\eta}{s_{m+1}} \int_{R^m} \frac{u(Z,1)dZ}{\{|X-Z|^2 + \eta^2\}^{(m+1)/2}} + \frac{2\eta}{s_{m+1}} \int_{R^m} \frac{u(Z,2\eta+1)dZ}{\{|X-Z|^2 + \eta^2\}^{(m+1)/2}}$$

$$\leqslant \frac{2}{s_{m+1}}\,\eta^{-m}(M(u,1) + M(u,2\eta+1))$$

$$= C_2, \text{ say.}$$

Also there is a constant C_1 such that $u(X,1) \leqslant C_1$ for all $X \in R^m$. Now put

$$H_\eta(X,y) = \frac{\eta + 1 - y}{\eta} C_1 + \frac{y - 1}{\eta} C_2,$$

where $(X,y) \in R^m \times [1,\eta+1]$. Then H_η is harmonic in $R^m \times (1,\eta+1)$. Also, $H_\eta = C_1 \geqslant u$ on $R^m \times \{1\}$, $H_\eta = C_2 \geqslant u$ on $R^m \times \{\eta+1\}$, and by (1)

$$\lim_{\substack{|(X,y)|\to+\infty\\ (X,y)\in R^m\times(1,\eta+1)}} \inf H_\eta(X,y) \geqslant 0 = \lim_{\substack{|(X,y)|\to+\infty\\ (X,y)\in R^m\times(1,\eta+1)}} u(X,y)$$

Thus

(7) $$H_\eta \geq u \quad \text{in} \quad R^m \times (1,\eta+1).$$

Let h_η be the least harmonic majorant of u in $R^m \times (1,\eta+1)$. If $\eta > 1$, we have

$$\begin{aligned} h_\eta(0,\ldots,0,2) &\leq H_\eta(0,\ldots,0,2) \\ &= \frac{\eta-1}{\eta} C_1 + \frac{1}{\eta} C_2 \\ &= O(1) + \frac{2}{s_{m+1}} \eta^{-m-1} M(u,2\eta+1) \quad \text{as} \quad \eta \to +\infty. \end{aligned} \tag{8}$$

Now $\lim_{\eta \to +\infty} H_\eta$ is either harmonic in $R^m \times (1,+\infty)$ or identical to $+\infty$ in $R^m \times (1,+\infty)$. Since u has no harmonic majorant in $R^m \times (1,+\infty)$, we must have $\lim_{\eta \to +\infty} h_\eta \to +\infty$. In particular, $h_\eta(0,\ldots,0,2) \to +\infty$ as $\eta \to +\infty$, and (8) now implies that $M(u,2\eta+1)/\eta^{m+1} \to +\infty$ as $\eta \to +\infty$. This proves the theorem.

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